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Algebras of finite global dimension

Dieter Happel and Dan Zacharia

Abstract Let Λ be a finite dimensional algebra over an algebraically closed field k. We survey some results on algebras of finite global dimension and address some open problems.

Let Λ be a finite dimensional algebra over an algebraically closed field k. We denote by $\operatorname{mod}\Lambda$ the category of finitely generated left Λ-modules. In this article we are mainly interested in algebras of finite global dimension, so each $X \in \operatorname{mod}\Lambda$ admits a finite projective resolution, or equivalently each simple Λ-module admits a finite projective resolution. It is well-known that the global dimension, $\operatorname{gldim}\Lambda$, of Λ is the maximum of the lengths of these finitely many minimal projective resolutions of the simple Λ-modules. These notions go back to the pioneering book by Cartan and Eilenberg [11]. They were intensively studied in a famous series of ten papers in the Nagoya Journal written by various authors and published over the years 1955 to 1958, see [1], [2], [8], [17], [18], [19], [20], [21], [40] and [41] for this series of articles.

The global dimension being preserved under Morita equivalence, implies that we may assume without loss of generality that Λ is basic. As the field k is algebraically closed, Λ is given by a quiver with relations. We briefly recall the construction. We start by recalling the definition of an admissible ideal. Let Q be a finite quiver and let

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kQ be the path algebra over k. Recall that the set $\mathscr{W} = \{\text{paths in } Q\}$ forms a $k-$basis for kQ. Denote by $Q_{\geq t}$ the two sided ideal of kQ generated by all the paths in Q of length t. A two sided ideal I in kQ is called *admissible* if there exists a natural number $t \geq 2$ such that

$$Q_{\geq t} \subseteq I \subseteq Q_{\geq 2}.$$

Then it is well-known (see [28]) that every basic finite dimensional k-algebra Λ satisfies $\Lambda \simeq kQ/I$ for some finite quiver Q and admissible ideal I in kQ. Note that the quiver Q is uniquely determined by Λ. By abuse of language, a minimal generating set of the ideal I is called a set of relations for Λ. This set is always finite, but we may have different choices for the relations for Λ. In particular there is usually no canonical choice for the relations. Note also that due to our assumptions on the admissible ideal I, the quiver Q can be recovered from $\Lambda = kQ/I$. To be more precise, the vertices of Q correspond to the (isomorphism classes of) simple Λ-modules and the number of arrows from a simple Λ-module S to a simple Λ-module S' coincides with $\dim_{\mathrm{k}} \operatorname{Ext}^1_\Lambda(S,S')$. We remind the reader that there is also a more ring theoretic version to find the quiver of Λ (see for example [5]).

We will address two basic questions for algebras of finite global dimension. First we deal with the question on possible obstructions for the quiver of an algebra of finite global dimension. It has been known for a long time [45], [36] that the quiver of an algebra Λ of finite global dimension does not contain any loops, or equivalently $\operatorname{Ext}^1_\Lambda(S,S) = 0$ for all simple $\Lambda-$modules S. This result has been recently strengthened by [37]. For details see Section 2.

There is a completely different behavior for algebras Λ of finite global dimension satisfying $\operatorname{gldim}\Lambda \leq 1$ or $\operatorname{gldim}\Lambda \geq 2$.

In the case when $\operatorname{gldim}\Lambda \leq 1$, the algebra Λ is hereditary. Thus $\Lambda = kQ$ for some finite quiver Q. Since we assume that Λ is finite dimensional, this implies that the quiver Q has no oriented cycles. Conversely, if Q does not contain any oriented cycle, then the algebra kQ satisfies $\operatorname{gldim} kQ \leq 1$. Of course $\operatorname{gldim} kQ = 0$ if and only if Q contains no arrows. So this case is well-understood.

In contrast to the first case the situation becomes more complicated and interesting for higher values of $\operatorname{gldim}\Lambda$. Moreover there are still a lot of open problems to deal with in this case. We will show in Section 3 that for an arbitrary quiver there will exist a two sided ideal I in the path algebra kQ such that for $\Lambda = kQ/I$ we have that $\operatorname{gldim}\Lambda \leq 2$, as long as the quiver does not contain a loop. This had been known for a long time by Dlab and Ringel [14] and [15]. It was recently rediscovered independently by N. Poettering [50] and the first author. We are grateful to C. M. Ringel for pointing out the two references mentioned above. We will also prove another related result in this section and pose a few open problems. For example, obtaining necessary and sufficient conditions on a given quiver Q such that there exists a two sided ideal I with $\operatorname{gldim} kQ/I = d$ for a prescribed natural number $d \geq 3$ is still open.

In Section 4 we address the following two related problems. Fix a quiver Q without loops, and consider the following set of algebras

$$\mathscr{A}(Q) = \{kQ/I \mid \dim_k kQ/I < \infty \text{ and gldim } kQ/I < \infty\}.$$

We then define the following

$$d(Q) = \sup\{\dim_k kQ/I \mid kQ/I \in \mathscr{A}(Q)\}$$

and

$$g(Q) = \sup\{\text{gldim} kQ/I \mid kQ/I \in \mathscr{A}(Q)\}.$$

The basic problem is whether or not these are finite. In general this seems to be an open problem. Note that the set $\mathscr{A}(Q)$ can be infinite. A concrete example is included in Section 4.

We will look at the relationship between $d(Q)$ and $g(Q)$. For example it follows from a result in [53] that $g(Q) < \infty$ if $d(Q) < \infty$. We will also discuss what is known about $d(Q)$ and $g(Q)$ when one restricts the algebras in $\mathscr{A}(Q)$ to some special subclasses of algebras. These subclasses include the serial, monomial and quasi-hereditary algebras. We remind the reader that except for a few instances, the global dimension of an algebra Λ does not depend solely on the number of simple Λ-modules, nor on the Loewy length of the algebra Λ. We refer to Section 4 for details and some examples.

We are aware of quite a number of other problems and results for algebras of finite global dimension. For example we could mention the Cartan determinant problem (see for example [54]). Also we will not deal with homologically finite subcategories in module categories for algebras of finite global dimension (see for example [3], [4] or [33]). Moreover, in order to keep the level of exposition as simple as possible, we will not attempt to formulate the results in the most general form available and will not use the language of derived categories (see for example [30]). Lastly, we will not attempt to be complete. We rather concentrate on the things which were mentioned before.

We denote the composition of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in a given category $\mathscr{K}$ by fg. The notation and terminology introduced here will be fixed throughout this article. For unexplained representation-theoretic terminology, we refer to [5], and [51].

1 Preliminaries

In this section we recall some further notation and definitions that will be used in the paper. Keeping the notation from the introduction, we assume that the finite dimensional algebra Λ is isomorphic to kQ/I for a finite quiver Q and an admissible two sided ideal I in the path algebra kQ. The set of vertices Q_0 of Q will be identified with the set $\{1,2,\ldots,n\}$. If α is an arrow of Q, we denote by $s(\alpha) \in Q_0$ and by $e(\alpha) \in Q_0$ the starting point of α (end point respectively). A path

$w = (i|\alpha_1, \ldots, \alpha_r|j)$ from i to j in Q is by definition a sequence of consecutive arrows $\alpha_1, \ldots, \alpha_r$ such that $s(\alpha_1) = i$, $e(\alpha_t) = s(\alpha_{t+1})$, for $1 \leq t < r$ and $e(\alpha_r) = j$. The corresponding element w in kQ is denoted by $\alpha_1 \ldots \alpha_r$ and we say that this path has length r. For a vertex i of Q we denote by $S(i)$ the corresponding simple $\Lambda-$module and by $P(i)$ its projective cover. If $X \in \operatorname{mod}\Lambda$ we denote by $\operatorname{proj.dim}_\Lambda X$ the projective dimension of X, that is, the length d of a minimal projective resolution of X:

$$0 \to P^d \to P^{d-1} \to \cdots \to P^1 \to P^0 \to X \to 0$$

Let us pause for a moment. Since in this article we are mainly interested in algebras of finite global dimension, we want to point out here a few additional facts.

First, assume that Λ is given as kQ/I and that its quiver Q contains no oriented cycle. In this case the global dimension of Λ is always finite and is bounded by the length of a maximal path in Q. In particular, $\operatorname{gldim}\Lambda \leq |Q_0| - 1$. So the more interesting algebras for us in this context are those whose quivers contain oriented cycles. By glueing techniques one could even assume that each vertex of Q lies on an oriented cycle. But we refrain from going into this here.

We should also make another remark which is useful to keep in mind, but will not be further investigated in this paper. Suppose that we have given a quiver Q and an ideal I with generating set $r_1, \ldots, r_t$ having the property that the coefficients occurring in r_j only belong to $\{1, -1\}$ for all $1 \leq j \leq t$. Then we may consider $\Lambda_k = kQ/I$ for different fields k. It is well-known that $\operatorname{gldim}\Lambda_k$ may depend on the characteristic of the field k. Examples for this phenomena can be found for instance in [12] and [39]. They were obtained in the following way. Start with a simplicial complex and consider the induced partial order. Then consider the corresponding incidence algebra. Then simplicial homology is related to the Hochschild cohomology of the incidence algebra [24]. There are well known examples that simplicial homology depends on the characteristic of the field of coefficients. This is used in [39] to construct examples where the global dimension depends on the characteristic of the field. For related investigations we also refer to [27].

For later purposes we will need the construction of the *standard modules* $\Delta(i)$ for $i \in Q_0$. For this let $\leq$ be a fixed partial order on the set of vertices $Q_0 = \{1, 2, \ldots, n\}$. For a vertex $i \in Q_0$ the module $\Delta(i)$ is the largest quotient of $P(i)$ with composition factors $S(j)$ for $j \geq i$. We say (see [52]) that Λ is *strongly quasi-hereditary* (with respect to the partial order $\leq$) if for every $i \in Q_0$ there is an exact sequence

$$0 \to \Omega(\Delta(i)) \to P(i) \to \Delta(i) \to 0$$

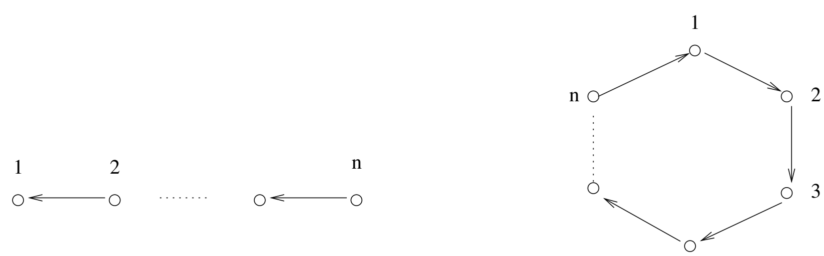
such that $\Omega(\Delta(i))$ is a direct sum of projective modules $P(j)$ with $j < i$ and $\operatorname{End}_\Lambda \Delta(i) \simeq k$ for all $1 \leq i \leq n$.

It was shown in [52] that a strongly quasi-hereditary algebra is quasi-hereditary in the usual sense (compare for example [13] or [16]) and that $\operatorname{gldim}\Lambda \leq n$, where n is the number of simple pairwise different Λ-modules. We refer to 4.7 for a bound on the global dimension for an arbitrary quasi-hereditary algebra Λ.

We will also need the concept of a monomial algebra. They are a good source of examples, since most homological problems are easy to decide. At the same time we would like to warn the reader that the monomial algebras are far from being typical amongst finite dimensional algebras. Recall that a basic finite dimensional k-algebra Λ is called a *monomial algebra* if $\Lambda \simeq kQ/I$ for a quiver Q and an admissible ideal I which can be generated by paths in Q. We point out that there is still no ring theoretic characterization for monomial algebras. The easiest examples of monomial algebras are the finite dimensional algebras whose quiver is a tree. For special homological properties of monomial algebras like the computations of minimal projective resolutions we refer to [26] and [35]. Note also that for monomial algebras, the global dimension does not depend on the ground field. For a path $w \in Q$ we denote by $\bar{w}$ the residue class of w in kQ/I. We have the following straightforward result whose proof is left to the reader.

Proposition 1.1. *Let $\Lambda = kQ/\langle w_1,\ldots,w_r\rangle$ be a finite dimensional monomial algebra. Then the set $\mathscr{J} = \{\bar{v} \in \Lambda \mid v \text{ a path in } Q, v \notin I\}$ is finite and forms a k-basis of Λ.* □

We recall the concept of a Nakayama algebra or generalized uniserial algebra. There are various possible definitions. We will use a quite concrete one. For different but equivalent formulations we refer for example to [5] or [44], [49]. Let Q be either a linearly oriented quiver with underlying graph $\mathbb{A}_n$ or a cycle $\tilde{\mathbb{A}}_n$ with cyclic orientation. So Q is one of the following



A quotient Λ of kQ by an admissible ideal is called a Nakayama algebra. Note that a Nakayama algebra is a special case of a monomial algebra.

In the first case of a linearly oriented $\mathbb{A}_n$ the corresponding Nakayama algebra Λ satisfies $\operatorname{gldim}\Lambda \le n-1$, so is always of finite global dimension, while in the second case of a cycle the corresponding Nakayama algebra may or may not be of finite global dimension, as very easy examples show. For example, let Q is a cyclic quiver with two vertices and arrows α,β. If we choose $I = \langle \alpha\beta\rangle$, then gldim $kQ/I = 2$. But if we choose $I' = \langle \alpha\beta,\beta\alpha\rangle$, then gldim $kQ/I' = \infty$. And of course both algebras are Nakayama algebras.

Given a Nakayama algebra Λ, one may associate to it its *Kupisch series*. From this series we can decide in a purely combinatorial way whether or not $\operatorname{gldim}\Lambda < \infty$. For details we refer to [22].

2 Obstructions

First we will recall the definition of the Hochschild homology and cohomology of a finite dimensional algebra Λ. We will not use the original definitions due to Hochschild [34], but use instead the characterization given in [11]. For this let $\Lambda^e = \Lambda \otimes_k \Lambda^{op}$ be the enveloping algebra, where we have denoted by Λ^{op} the opposite algebra. Then the $\Lambda - \Lambda$-bimodule ${}_\Lambda\Lambda_\Lambda$ is considered as a Λ^e-module. For $i \geq 0$ the i-th Hochschild homology space is denoted by $H_i(\Lambda)$ and is defined as $\mathrm{Tor}_i^{\Lambda^e}(\Lambda,\Lambda)$. We denote by $[\Lambda,\Lambda]$ the k-subspace of Λ generated by the commutators, thus elements of the form $\lambda\mu - \mu\lambda$ for $\lambda,\mu \in \Lambda$. Then it is easy to see that $H_0(\Lambda)$ coincides with the factor space $\Lambda/[\Lambda,\Lambda]$.

Also for $i \geq 0$ we denote by $H^i(\Lambda)$ the i-th Hochschild cohomology space and this is defined as $\mathrm{Ext}^i_{\Lambda^e}(\Lambda,\Lambda)$. So $H^0(\Lambda)$ coincides with the center of Λ and $H^1(\Lambda)$ is the factor space of all derivations on Λ by the subspace of inner derivations.

The following theorem is due to Keller [42].

Theorem 2.1. *Let Λ be a finite dimensional algebra of finite global dimension. Then $H_i(\Lambda) = 0$ for $i > 0$ and $H_0(\Lambda) = k^{|Q_0|}$. In particular, $\Lambda/[\Lambda,\Lambda] = k^{|Q_0|}$.*

We remark that in [29] it is conjectured that the converse of 2.1 holds. There this converse is proved in special cases such as monomial algebras.

We point out that there is no characterization of algebras of finite global dimension through Hochschild cohomology. It is easy to see that for an algebra Λ of finite global dimension d the Hochschild cohomology spaces $H^j(\Lambda) = 0$ for $j > d$ (see for example [32]), in particular the cohomology algebra $H^*(\Lambda) = \bigoplus_{j=0}^{\infty} H^i(\Lambda)$ is finite dimensional over k. But this does not yield a characterization, since the converse does not hold as shown in [10]. For the convenience of the reader we will include their example. For $q \in k$ consider the finite dimensional algebra $\Lambda_q = k\langle x,y\rangle/\langle x^2, xy+qyx, y^2\rangle$, where $k\langle x,y\rangle$ is the free algebra in two generators. It is easy to see that Λ_q is a selfinjective algebra so $\mathrm{gldim}\,\Lambda_q = \infty$. If q is not a root of unity, then it is shown in [10] that $\dim_k H^*(\Lambda) = 5$.

We will need the following observation: Let $\Lambda = kQ/I$. We denote the primitive orthogonal idempotents of Λ corresponding to the vertices $\{1,\ldots,n\}$ of Q by $e_1,\ldots,e_n$. Let $w = (i|\alpha_1,\ldots,\alpha_r|j)$ be a path in Q, and assume that $i \neq j$. Then $\bar{w} = e_i\bar{w} = e_i\bar{w} - \bar{w}e_i$, since $\bar{w}e_i = 0$ if $i \neq j$. Therefore $\bar{w} \in [\Lambda,\Lambda]$ in this case. Clearly, $e_i \notin [\Lambda,\Lambda]$, for $1 \leq i \leq n$. If we assume in addition that Λ has finite global dimension, we have more: In this case, since $H_0(\Lambda)$ is spanned as a vector space by the residue classes of $e_1,\ldots,e_n$, every proper cycle w of Q has the property that its residue class $\bar{w}$ in Λ belongs to $[\Lambda,\Lambda]$.

The following result was first shown in [45] and reproved in [36]. This theorem is usually referred to as the no-loop conjecture. (Actually a stronger result is shown in [45]. We refer to this paper for details.)

Theorem 2.2. *Let $\Lambda = kQ/I$ be a finite dimensional algebra such that Q contains a loop. Then* $\mathrm{gldim}\,\Lambda = \infty$.

Proof. We give a sketch of the proof, and we use 2.1. Assume to the contrary that Λ is of finite global dimension and let α be the loop in Q. Let $e_1,\dots,e_n \in \Lambda$ be the elements corresponding to the vertices of Q. By the previous remarks, the residue classes of $\bar{e}_1,\dots,\bar{e}_n$ form a basis of $\Lambda/[\Lambda,\Lambda]$, and $\bar{\alpha} \in [\Lambda,\Lambda]$. But it is readily checked that this yields a contradiction, hence $\operatorname{gldim}\Lambda = \infty$.

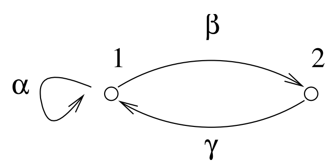
There is a local version of the no-loop conjecture called the strong no-loop conjecture. This was only proved recently in [37] and is formulated as follows.

Theorem 2.3. *Let $\Lambda = kQ/I$ be a finite dimensional algebra such that Q contains a loop at the vertex i. Then the simple module $S(i)$ has infinite projective dimension.*

A more general version of the strong no-loop conjecture is still open and is called the extension conjecture in [37].

Extension conjecture *Let $\Lambda = kQ/I$ be a finite dimensional algebra such that Q contains a loop at the vertex i, then $\operatorname{Ext}^j_\Lambda(S(i),S(i)) \neq 0$ for infinitely many j.*

We point out that the more general question, whether $\operatorname{Ext}^1_\Lambda(S,S) \neq 0$ for a Λ-module S implies $\operatorname{Ext}^j_\Lambda(S,S) \neq 0$ for all j, has a negative answer. For this consider the algebra Λ given by the following quiver with relations (see also [27])



and relations $\alpha^2 - \beta\gamma, \gamma\alpha\beta, \gamma\beta$. Then the minimal projective resolution of $S(1)$ is easily computed as

$$\cdots \to P(2) \to P(1) \to P(1) \oplus P(2) \to P(1) \to S(1) \to 0$$

and $\Omega^4 S(1) = S(1)$. So $\operatorname{Ext}^j_\Lambda(S(1),S(1)) = 0$ for $j \equiv 3 \bmod 4$. But the extension conjecture clearly holds in this example.

We refer to [27] and [47] for some classes of algebras where the stronger version of the extension conjecture will hold.

3 Constructions

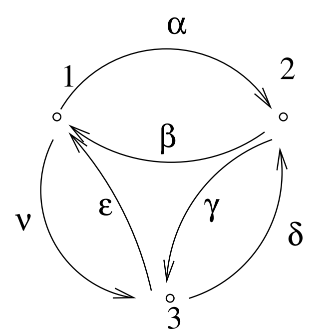
In the previous section we have seen that one obstruction on Q for being the quiver of an algebra of finite global dimension was the existence of a loop in Q. Here we will show that this is the only obstruction. We are grateful to C. M. Ringel for pointing out to us that this construction had already been carried out in [14] and also in [15] even in the more general setting of a species. We will come back to those papers also in the next section. For the convenience of the reader we give the construction of such an algebra of finite global dimension starting from an arbitrary quiver without loops, but refer for a detailed proof to [14], [15] or [50]. In general there will be several such algebras of finite global dimension having the same quiver. This will be the subject of the next section.

Theorem 3.1. *Let Q be a quiver without loops. Then there exists an admissible ideal I such that* $\operatorname{gldim} kQ/I \leq 2$.

Proof. We only give a sketch of the proof. Let $Q_0 = \{1, \ldots, n\}$ be the set of vertices of Q. For $2 \leq i \leq n$ let α_{ij_i} be the arrows of Q such that $e(\alpha_{ij_i}) = i$, $s(\alpha_{ij_i}) < i$ and β_{im_i} the arrows of Q such that $s(\beta_{im_i}) = i$, $e(\beta_{im_i}) < i$. Let I be the two sided ideal in kQ generated by $\alpha_{ij_i}\beta_{im_i}$ for $2 \leq i \leq n$ and all j_i, m_i. Then I is an admissible ideal and the algebra $\Lambda = kQ/$ is a finite dimensional monomial algebra. Using [26] one can easily show that $\operatorname{gldim}\Lambda \leq 2$.

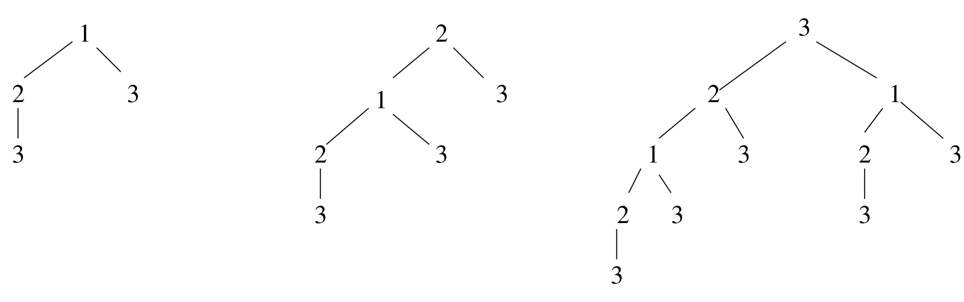
One can even show that the algebra constructed in the proof is strongly quasi-hereditary in the sense of Ringel. For a definition we refer to Section 1.

We illustrate the previous theorem with a specific example. We will use the following notation. The first set of arrows occurring in the sketch of the proof of 3.1 is denoted by α_{i*} while the second is denoted by β_{i*} for a vertex i of Q. Let Q be the following quiver:



Thus $\alpha_{2*} = \{\alpha\}, \beta_{2*} = \{\beta\}$ and $\alpha_{3*} = \{\nu, \gamma\}, \beta_{3*} = \{\delta, \varepsilon\}$.
Let $I = \langle \alpha\beta, \gamma\delta, \gamma\varepsilon, \nu\delta, \nu\varepsilon \rangle$ and $\Lambda = kQ/I$.

Then one can read the indecomposable projective Λ-modules and their Loewy series of from the following diagrams:



The following exact sequences are the minimal projective resolutions of the simple Λ-modules

$$0 \to P(1)^2 \oplus P(2) \to P(2) \oplus P(3) \to P(1) \to S(1) \to 0$$
$$0 \to P(1) \oplus P(2) \to P(3) \oplus P(1) \to P(2) \to S(2) \to 0$$
$$0 \to P(1) \oplus P(2) \to P(3) \to S(3) \to 0.$$

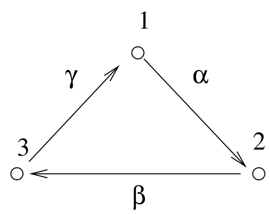
The standard modules are given by

$$\Delta(1) = P(1), \Delta(2) = P(2)/P(1) \,\text{and}\, \Delta(3) = S(3).$$

One may ask even more detailed questions, such as the existence of an admissible ideal I in a given quiver Q such that the global dimension of kQ/I is a prescribed natural number. Trivially one needs more conditions. For example if there is no path of length two, then gldim $kQ/I \leq 1$. Some answers are given in [50], but the precise nature of the needed conditions remains unclear to us. For example the following can be shown (see for example [50]).

Theorem 3.2. *Let Q be a quiver without loops containing a path of length d that consists of d pairwise distinct arrows. Then there exists an admissible ideal I generated by paths such that* gldim $kQ/I = d$.

We point out that the converse is not true. Consider the algebra Λ given by the cyclic quiver Q on three vertices $1,2,3$ and arrows α, β, γ.



Choose as admissible ideal of kQ the ideal $I = \langle \gamma\alpha\beta, \alpha\beta\gamma\alpha \rangle$. Then it is a straightforward computation to show that the following are minimal projective resolutions of the simple modules over kQ/I:

$$0 \to P(3) \to P(2) \to P(2) \to P(1) \to S(1) \to 0$$
$$0 \to P(3) \to P(2) \to S(2) \to 0$$
$$0 \to P(3) \to P(2) \to P(3) \to P(1) \to P(3) \to S(3) \to 0.$$

We have gldim $kQ/I = 4$, but there is no path in Q consisting of four pairwise different arrows. However, we can say the following:

Proposition 3.3. *Let Q be a quiver without loops and $\Lambda = kQ/I$ be a finite dimensional algebra with* gldim $\Lambda = d < \infty$. *Then there exists a path of length d in Q.*

Proof. Assume to the contrary that Q does not contain a path of length d. Then the length of all the paths in Q is bounded by $d-1$. In particular we conclude that Q does not contain an oriented cycle, hence is directed. But we have remarked before in Section 1 that then $\operatorname{gldim}\Lambda \leq d-1$, a contradiction. So Q contains a path of length d.

We present now another result in the spirit of 3.1.

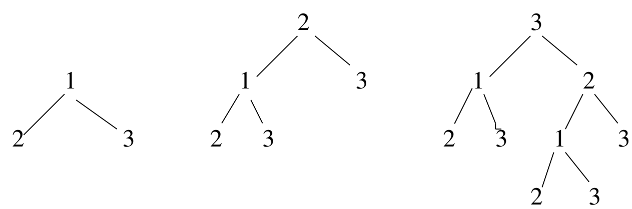
Theorem 3.4. *Let Q be a quiver without loops containing n vertices. Then there exists a monomial algebra $\Lambda = kQ/I$ with* $\operatorname{gldim}\Lambda \leq n$.

Proof. Let $Q_0 = \{1,\dots,n\}$ be the set of vertices of Q. For each vertex $1 \leq i < n$ consider the following set of arrows:

$$\{\alpha_{ij} \,|\, s(\alpha_{ij}) = i \text{ and } e(\alpha_{ij}) > i\}$$

If β is an arrow with $s(\beta) = e(\alpha_{ij})$ for some α_{ij}, then $\alpha_{ij}\beta \in I$. This defines Λ. It is easy to verify that $\operatorname{proj.dim}_\Lambda S(i) \leq n-i+1$ for $1 \leq i \leq n$, so $\operatorname{gldim}\Lambda \leq n$.

We consider an example. Let Q be the quiver from the example following 3.1. Then $I = \langle \alpha\gamma, \alpha\beta, \nu\delta, \nu\varepsilon, \gamma\delta, \gamma\varepsilon \rangle$. The indecomposable projective Λ-modules and their Loewy series can be read from the following diagrams:



The standard modules are again given by

$$\Delta(1) = P(1), \Delta(2) = P(2)/P(1) \text{ and } \Delta(3) = S(3).$$

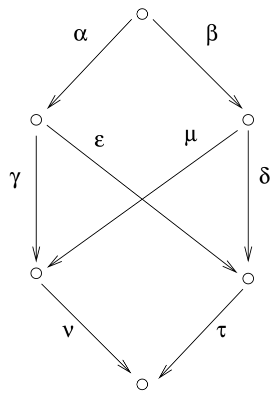
Note that also in this case we obtain a strongly quasi-hereditary algebra.

4 Bounds

Unless otherwise stated, Q will denote in this section a quiver with no loops. We will deal with the following set of algebras:

$$\mathscr{A}(Q) = \{kQ/I \,|\, \dim_k kQ/I < \infty \text{ and } \operatorname{gldim} kQ/I < \infty\}$$

By 3.1 we know that $\mathscr{A}(Q) \neq \emptyset$. We point out that $\mathscr{A}(Q)$ usually will contain an infinite number of non-isomorphic algebras as the following example from [9] shows. Consider the following quiver Q



and $I_q = \langle \alpha\varepsilon - \beta\delta, \alpha\gamma - \beta\mu, \mu\nu - \delta\tau, \gamma\nu - q\varepsilon\tau \rangle$ for some $q \in k$. Set $\Lambda_q = kQ/I_q$. Then $\operatorname{gldim}\Lambda_q \leq 2$ and Λ_q defines an infinite family of finite dimensional algebras, which all have the same k-dimension.

We then define the following

$$d(Q) = \sup\{\dim_k kQ/I \,|\, kQ/I \in \mathscr{A}(Q)\}$$

and

$$g(Q) = \sup\{\operatorname{gldim} kQ/I \,|\, kQ/I \in \mathscr{A}(Q)\}.$$

The next result from [53] yields a relationship between $d(Q)$ and $g(Q)$. Its proof uses an upper semi-continuity argument on the algebraic variety of finite dimensional algebras of a fixed k-dimension.

Theorem 4.1. *There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such* $\operatorname{gldim}\Lambda \leq f(d)$ *for all finite dimensional algebras Λ with* $\dim_k \Lambda \leq d$ *and* $\operatorname{gldim}\Lambda < \infty$.

Corollary 4.2. *$g(Q) < \infty$ if $d(Q) < \infty$.*

We will address the following three questions.

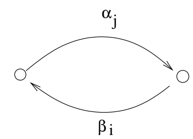
Questions 4.3 *Let Q be a quiver without loops.*

(1)Is $d(Q) < \infty$?
(2)Is the converse of 4.2 true?
(3)Is $\operatorname{gldim} kQ/I \leq \dim_k kQ/I$ *in case* $\operatorname{gldim} kQ/I < \infty$?

In general we do not have an answer to these questions. We will now survey what is known for special classes of algebras. But before, we would like to remind the reader of two classes of examples which show that the values of the global dimension cannot only depend on the number of simple modules nor on the Loewy length of the algebra. Recall that the Loewy length of Λ is the least positive integer d such that $\operatorname{rad}^d\Lambda = 0$, where we have denoted by $\operatorname{rad}\Lambda$ the radical of Λ. The following two examples motivated us to consider $\mathscr{A}(Q)$ for a fixed quiver Q.

Example 4.4. The first example is due to E. Green [25], but see also [31]. It deals with the question of dependence of the global dimension on the number of simple modules.

For each natural number n, let Q_n be given by



for $1 \leq i, j \leq n$.
Let $I_n = \langle \beta_i \alpha_j, 1 \leq i \leq n, 1 \leq j \leq i, \alpha_j \beta_i, 2 \leq j \leq n, 1 \leq i \leq j-1 \rangle$.
Set $\Lambda_n = kQ_n/I_n$. Then $\operatorname{gldim} \Lambda_n = 2n$. Note that $\operatorname{rad}^{2n} \Lambda_n \neq 0$.

Example 4.5. The next example is due to E. Kirkman and J. Kuzmanovich and deals with the dependence of the global dimension on the Loewy length.

Let $n \geq 1$ and let the quiver Q_n be given as in the previous example.
Set $I_n = \langle \beta_i \alpha_j \beta_l$ for $1 \leq i, j, l \leq n, \alpha_i \beta_{i+l} - \alpha_{i+l} \beta_{i+l}$ for $l \geq 1, \alpha_i \beta_j$ for $i > j, \beta_i \alpha_i$ for $1 \leq i \leq n \rangle$.

Let $\Lambda_n = kQ_n/I_n$. Then $\operatorname{gldim} \Lambda_n = 2n+1$. Note that $\operatorname{rad}^4 \Lambda_n = 0$.

We will now mention some results where positive answers to the questions above have been obtained for special classes of algebras.

The first one is due to W. Gustafson [28] and deals with Nakayama algebras.

Theorem 4.6. *Let Λ be a Nakayama algebra of finite global dimension with n non isomorphic simple modules. Then* $\operatorname{gldim} \Lambda \leq 2n-2$. *Moreover, the Loewy length of Λ is bounded by* $2n-1$.

Proof. We give a proof of the bound of the global dimension different from the one given in [28] and refer for the bound of the Loewy length to [28]. We will proceed by induction on n. Trivially we may restrict to the case of a cyclic quiver. For $n = 2$ it is easy to see that the only Nakayama algebra with two simple modules is given by the Auslander algebra of $k[t]/\langle t^2 \rangle$, hence has global dimension two. So the result holds for $n = 2$. Since Λ has finite global dimension, there is a simple Λ-module, say $S(n)$ of projective dimension 1 (see [55]). Let $P = \bigoplus_{i=1}^{n-1} P(i)$ and let $\Gamma = \operatorname{End}_\Lambda P$. Then clearly Γ is again a Nakayama algebra and by [54], $\operatorname{gldim} \Gamma \leq \operatorname{gldim} \Lambda < \infty$. So by induction we have that $\operatorname{gldim} \Gamma \leq 2(n-1)-2$, since Γ has $n-1$ simple modules. The simple Γ-modules are all of the form $\operatorname{Hom}_\Lambda(P,S)$ where S is a simple Λ-module such that S is not isomorphic to $S(n)$. If $S \ncong S(n)$ is such a simple Λ-module and

$$0 \to P^t \to P^{t-1} \to \cdots \to P^1 \to P(S) \to S \to 0$$

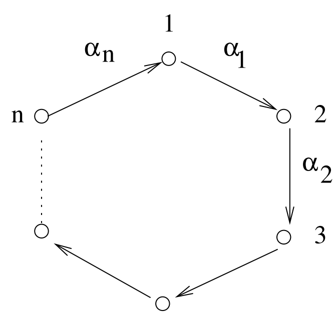
is a minimal projective resolution of S, then all P^i are indecomposable. Moreover

$$(*)\ 0 \to \operatorname{Hom}(P,P^t) \to \cdots \to \operatorname{Hom}(P,P(S)) \to \operatorname{Hom}(P,S) \to 0$$

is a projective resolution of the simple Γ-module $\mathrm{Hom}(P,S)$. The resolution $(*)$ is usually not minimal. This happens if $\mathrm{rad}\,P(n) \to P(n)$ occurs in $(*)$ and the map is the standard embedding. But this map can only occur in $(*)$ at the end of the resolution, so we can conclude that $\mathrm{gldim}\,\Gamma \geq \mathrm{gldim}\,\Lambda - 2$. So by induction $\mathrm{gldim}\,\Lambda \leq 2n-2$.

The result 4.6 may be reformulated as follows. Let Q be an oriented cycle with $n \geq 2$ vertices. Then $g(Q) \leq 2n-2$.

We include now an example from [28] showing that the above bound is optimal, that is $g(Q) = 2n-2$ for Q an oriented cycle with $n \geq 2$ vertices. Let Q be an oriented cycle with n vertices and n arrows α_i for $1 \leq i \leq n$ such that $s(\alpha_i) = i = e(\alpha_{i-1})$ for $1 < i < n$ and $e(\alpha_n) = 1 = s(\alpha_1)$.



Let $w_1 = \alpha_1 \dots \alpha_n$ and for $2 \leq i \leq n-1$ let $w_i = \alpha_i \alpha_{i+1} \dots \alpha_n \alpha_1 \dots \alpha_i$. Then let $\Lambda = kQ/\langle w_1, \dots, w_{n-1}\rangle$. Then $\mathrm{gldim}\,\Lambda = 2n-2$. In fact, it is readily checked that $\mathrm{proj.dim}_\Lambda S(n-i) = 2i+1$ for $0 \leq i < n-1$ and $\mathrm{proj.dim}_\Lambda S(1) = 2n-2$.

We point out that the second bound in 4.6 is also optimal. For more details we refer to [28].

Contrary to the general situation, the global dimension for quasi-hereditary algebras is bounded by a function on the number of simple modules. We have the following result from [16].

Theorem 4.7. *If Λ is a quasi-hereditary algebra with n non isomorphic simple modules, then* $\mathrm{gldim}\,\Lambda \leq 2n-2$.

Results in [14] and [15] may also be reformulated. Let

$$\mathscr{A}'(Q) = \{kQ/I \mid \dim_k kQ/I < \infty \text{ and } kQ/I \text{ is quasi-hereditary}\}.$$

and

$$d'(Q) = \sup\{\dim_k kQ/I \mid kQ/I \in \mathscr{A}'(Q)\}.$$

Note that a quasi-hereditary algebra is always of finite global dimension (see for example [16]), so $\mathscr{A}'(Q) \subset \mathscr{A}(Q)$.

Theorem 4.8. $d'(Q) < \infty$.

Corollary 4.9. *Let Q be a quiver without loops. Then*

$$\sup\{\dim_k kQ/I \mid \operatorname{gldim} kQ/I \le 2\} \in \mathbb{N}.$$

Proof. It is well-known that any algebra of global dimension at most two is quasi-hereditary [16]. So the result follows from 4.8.

For monomial algebras we obtain an affirmative answer for our question 3 from [38]. Actually a stronger statement is shown in [38]. We refer to this article for further details.

Theorem 4.10. *Let Λ be a finite dimensional monomial algebra of finite global dimension. Then* $\operatorname{gldim}\Lambda \le \dim_k \Lambda$.

One may restrict question one to subclasses of algebras, for example monomial algebras. But even then this is an open problem. The results and examples below show that question one is related to Section 2 in the sense that there are obstructions on the admissible ideal I of an algebra of finite global dimension of the form kQ/I for a quiver Q without loops.

In the case of a monomial algebra $\Lambda = kQ/I$ we can say the following.

Let $w \in Q$ be an oriented cycle, say $w = \alpha_1 \dots \alpha_t$. Then for $1 \le i \le t$ we set the "rotated" cycle $w_i = \alpha_i \dots \alpha_t \alpha_1 \dots \alpha_{i-1}$.

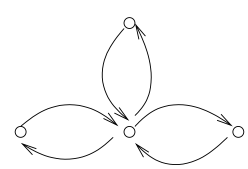
Proposition 4.11. *Let $\Lambda = kQ/I$ be a monomial algebra of finite global dimension. Then for each oriented cycle w of Q of length t there is $1 \le i \le t$ such that $w_i \in I$.*

Proof. Let w be an oriented cycle of Q of length t. Without loss of generality we may assume that w is a nonzero path in Λ, that is $w \notin I$. Then by 2.1 we infer that $w \in [\Lambda, \Lambda]$ since the image of w is 0 in $H_0(\Lambda)$. Thus there are elements $u_i, v_i \in \Lambda$ for $1 \le i \le m$ such that $w = \sum_{i=1}^m u_i v_i - v_i u_i$. Since Λ is monomial the set of nonzero paths (including the paths of length 0) in Λ forms a k-basis of Λ (compare 1.1), so we may also assume that the u_i and v_i are nonzero paths in Λ. It follows that $w = u_i v_i$ for some i and that $v_i u_i \in I$. But $v_i u_i$ is clearly a rotation of w.

Corollary 4.12. *Let $\Lambda = kQ/I$ be a monomial algebra of finite global dimension. Then for each oriented cycle w of Q we have that $w^2 \in I$.*

Proof. If $w \in Q$ is an oriented cycle, then w^2 contains any rotation as a subpath, hence $w^2 \in I$ by 4.11.

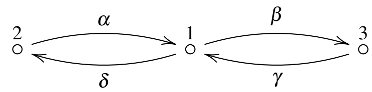
We remark that 4.12 does not imply an affirmative answer to question 1 for monomial algebras. If Q is the quiver



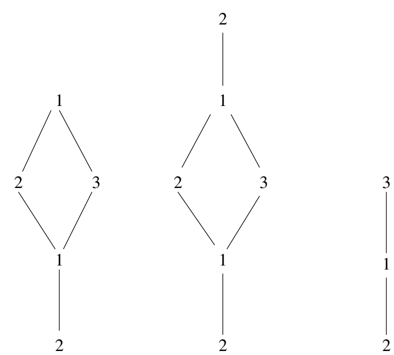
then there exist paths in Q of arbitrary length which do not contain the square of a cycle in Q. This follows from [46]. See also [48].

At the same time we do not know whether 4.11 is sufficient for yielding an affirmative answer to question 1 for the class of monomial algebras.

We would like to remark that 4.12 will usually fail for non monomial algebras as the following example shows. Let Λ be given by the following quiver Q



and the ideal I generated by $\delta\alpha - \beta\gamma$ and $\gamma\beta$. Then the indecomposable projective Λ-modules and their Loewy series are given by the following diagrams:



The minimal projective resolutions of the simple Λ-modules are:

$$0 \to P(1) \to P(2) \oplus P(3) \to P(1) \to S(1) \to 0$$

$$0 \to P(1) \to P(2) \to S(2) \to 0$$

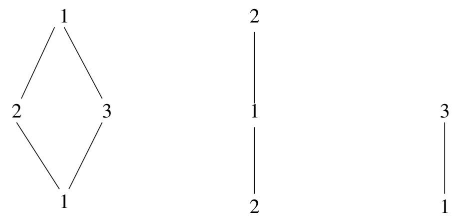
$$0 \to P(3) \to P(1) \to P(3) \to S(3) \to 0,$$

hence $\mathrm{gl.dim}\Lambda = 2$. Consider the cycle $w = \alpha\delta$. Then $w^2 \notin I$. Note however, that the rotation $\delta\alpha$ of w is in I.

We end with a discussion of the following result by [7]. First we recall the definition of a truncated m-cycle. Let Q be a quiver, I an admissible ideal and let $m \in \mathbb{N}$. A cycle $w \in Q$ is called a *truncated m-cycle*, if the composition of any m composable arrows in w belongs to I and the composition of any $m-1$ composable arrows of w does not belong to I. The following is shown in [7].

Theorem 4.13. *Let* $\Lambda = kQ/I$. *If* Q *contains a truncated* 2*-cycle, then* $\mathrm{gldim}\Lambda = \infty$.

One of the questions asked in [7] is whether the truncated 2-cycle condition in 4.13 could be replaced by a truncated m-cycle condition for $m \geq 3$. The answer is negative, as the following example shows. Indeed consider again the quiver Q from the previous example. Let $I = \langle \delta\alpha - \beta\gamma, \alpha\beta, \gamma\beta, \gamma\delta \rangle$ and $\Lambda = kQ/I$. The indecomposable projective Λ-modules are given as follows



It is easy to see that $\operatorname{gldim}\Lambda = 4$. In fact, the minimal projective resolutions of the simple Λ-modules are given as follows

$$0 \to P(3) \to P(1) \to P(2) \oplus P(3) \to P(1) \to S(1) \to 0,$$

$$0 \to P(3) \to P(1) \to P(2) \to S(2) \to 0$$

$$0 \to P(3) \to P(1) \to P(2) \oplus P(3) \to P(1) \to P(3) \to S(3) \to 0.$$

Now $\alpha\delta$ is a truncated 3-cycle, since $\alpha\delta\alpha, \delta\alpha\delta \in I$ and $\alpha\delta \notin I$ as well as $\delta\alpha \notin I$.

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